HANOI MATHEMATICAL SOCIETY

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HANOI OPEN MATHEMATICAL OLYMPIAD PROBLEMS

HANOI - 2011

Contents

Chapter 1

Hanoi Open Mathematical Olympiad

1.1 Hanoi Open Mathematical Olympiad 2006

1.1.1 Junior Section

Question 1. What is the last two digits of the number

$$(11 + 12 + 13 + \cdots + 2006)^2?$$

Question 2. Find the last two digits of the sum

$$2005^{11} + 2005^{12} + \cdots + 2005^{2006}.$$

Question 3. Find the number of different positive integer triples (x, y, z) satisfying the equations

$$x^2 + y - z = 100 \quad \text{and} \quad x + y^2 - z = 124.$$

Question 4. Suppose x and y are two real numbers such that

$$x + y - xy = 155 \quad \text{and} \quad x^2 + y^2 = 325.$$

Find the value of $\quad |x^3 - y^3|$.

Question 5. Suppose n is a positive integer and 3 arbitrary numbers are choosen from the set $\{1, 2, 3, \ldots, 3n+1\}$ with their sum equal to $3n + 1$.

What is the largest possible product of those 3 numbers?

Question 6. The figure $ABCDEF$ is a regular hexagon. Find all points M belonging to the hexagon such that

$$\text{Area of triangle } MAC = \text{Area of triangle } MCD.$$

Question 7. On the circle (O) of radius 15cm are given 2 points A, B. The altitude OH of the triangle OAB intersect (O) at C. What is AC if $AB = 16cm$?

Question 8. In ΔABC, $PQ//BC$ where P and Q are points on AB and AC respectively. The lines PC and QB intersect at G. It is also given $EF//BC$, where $G \in EF$, $E \in AB$ and $F \in AC$ with $PQ = a$ and $EF = b$. Find value of BC.

Question 9. What is the smallest possible value of

$$x^2 + y^2 - x - y - xy?$$

1.1.2 Senior Section

Question 1. What is the last three digits of the sum

$$11! + 12! + 13! + \cdots + 2006!$$

Question 2. Find the last three digits of the sum

$$2005^{11} + 2005^{12} + \cdots + 2005^{2006}.$$

Question 3. Suppose that

$$a^{\log_b c} + b^{\log_c a} = m.$$

Find the value of

$$c^{\log_b a} + a^{\log_c b}?$$

Question 4. Which is larger

$$2^{\sqrt{2}}, \quad 2^{1+\frac{1}{\sqrt{2}}} \quad \text{and} \quad 3.$$

Question 5. The figure $ABCDEF$ is a regular hexagon. Find all points M belonging to the hexagon such that

$$\text{Area of triangle } MAC = \text{Area of triangle } MCD.$$

Question 6. On the circle of radius 30cm are given 2 points A, B with $AB = 16$cm and C is a midpoint of AB. What is the perpendicular distance from C to the circle?

Question 7. In ΔABC, $PQ//BC$ where P and Q are points on AB and AC respectively. The lines PC and QB intersect at G. It is also given $EF//BC$, where $G \in EF$, $E \in AB$ and $F \in AC$ with $PQ = a$ and $EF = b$. Find value of BC.

Question 8. Find all polynomials $P(x)$ such that

$$P(x) + Pig(\frac{1}{x}ig) = x + \frac{1}{x}, \quad \forall x \neq 0.$$

Question 9. Let x, y, z be real numbers such that $x^2+y^2+z^2 = 1$. Find the largest possible value of

$$|x^3 + y^3 + z^3 - xyz|?$$

1.2 Hanoi Open Mathematical Olympiad 2007

1.2.1 Junior Section

Question 1. What is the last two digits of the number

$$(3 + 7 + 11 + \cdots + 2007)^2?$$

(A) 01; (B) 11; (C) 23; (D) 37; (E) None of the above.

Question 2. What is largest positive integer n satisfying the following inequality:

$n^{2006} < 7^{2007}$?

(A) 7; (B) 8; (C) 9; (D) 10; (E) 11.

Question 3. Which of the following is a possible number of diagonals of a convex

polygon?

(A) 02; (B) 21; (C) 32; (D) 54; (E) 63.

Question 4. Let m and n denote the number of digits in 2^{2007} and 5^{2007} when

expressed in base 10. What is the sum $m + n$?

(A) 2004; (B) 2005; (C) 2006; (D) 2007; (E) 2008.

Question 5. Let be given an open interval $(\alpha; eta)$ with $eta - \alpha = \frac{1}{2007}$. Determine the

maximum number of irreducible fractions $\frac{a}{b}$ in $(\alpha; eta)$ with $1 \leq b \leq 2007$?

(A) 1002; (B) 1003; (C) 1004; (D) 1005; (E) 1006.

Question 6. In triangle ABC, $\angle BAC = 60^0$, $\angle ACB = 90^0$ and D is on BC. If AD

bisects $\angle BAC$ and $CD = 3cm$. Then DB is

(A) 3; (B) 4; (C) 5; (D) 6; (E) 7.

Question 7. Nine points, no three of which lie on the same straight line, are located

inside an equilateral triangle of side 4. Prove that some three of these

points are vertices of a triangle whose area is not greater than $\sqrt{3}$.

Question 8. Let a, b, c be positive integers. Prove that

$$\frac{(b+c-a)^2}{(b+c)^2+a^2} + \frac{(c+a-b)^2}{(c+a)^2+b^2} + \frac{(a+b-c)^2}{(a+b)^2+c^2} \geq \frac{3}{5}.$$

Question 9. A triangle is said to be the Heron triangle if it has integer sides and

integer area. In a Heron triangle, the sides a, b, c satisfy the equation

$b = a(a - c)$. Prove that the triangle is isosceles.

Question 10. Let a, b, c be positive real numbers such that $\frac{1}{bc} + \frac{1}{ca} + \frac{1}{ab} \geq 1$. Prove

that $\frac{a}{bc} + \frac{b}{ca} + \frac{c}{ab} \geq 1$.

Question 11. How many possible values are there for the sum $a + b + c + d$ if a, b, c, d

are positive integers and $abcd = 2007$.

Question 12. Calculate the sum

$$\frac{5}{2.7} + \frac{5}{7.12} + \cdots + \frac{5}{2002.2007}.$$

Question 13. Let be given triangle ABC. Find all points M such that

area of ΔMAB= area of ΔMAC.

Question 14. How many ordered pairs of integers (x, y) satisfy the equation

$$2x^2 + y^2 + xy = 2(x + y)?$$

Question 15. Let $p = \overline{abc}$ be the 3-digit prime number. Prove that the equation

$$ax^2 + bx + c = 0$$

has no rational roots.

1.2.2 Senior Section

Sunday, 15 April 2007 09h00-11h00

Question 1. What is the last two digits of the number

$$\left(11^2 + 15^2 + 19^2 + \cdots + 2007^2\right)^2?$$

(A) 01; (B) 21; (C) 31; (D) 41; (E) None of the above.

Question 2. Which is largest positive integer n satisfying the following inequality:

$$n^{2007} > (2007)^n.$$

(A) 1; (B) 2; (C) 3; (D) 4; (E) None of the above.

Question 3. Find the number of different positive integer triples (x, y, z) satsfying the equations

$$x + y - z = 1 \quad \text{and} \quad x^2 + y^2 - z^2 = 1.$$

(A) 1; (B) 2; (C) 3; (D) 4; (E) None of the above.

Question 4. List the numbers $\sqrt{2}$, $\sqrt[3]{3}$, , $\sqrt[4]{4}$, $\sqrt[5]{5}$ and $\sqrt[6]{6}$ in order from greatest to
least.

Question 5. Suppose that A, B, C, D are points on a circle, AB is the diameter, CD
is perpendicular to AB and meets AB at E, AB and CD are integers
and $AE - EB = \sqrt{3}$. Find AE?

Question 6. Let $P(x) = x^3 + ax^2 + bx + 1$ and $|P(x)| \leq 1$ for all x such that $|x| \leq 1$.
Prove that $|a| + |b| \leq 5$.

Question 7. Find all sequences of integers $x_1, x_2, \ldots, x_n, \ldots$ such that ij divides
$x_i + x_j$ for any two distinct positive integers i and j.

Question 8. Let ABC be an equilateral triangle. For a point M inside ΔABC,
let D, E, F be the feet of the perpendiculars from M onto BC, CA, AB,
respectively. Find the locus of all such points M for which $\angle FDE$ is a
right angle.

Question 9. Let $a_1, a_2, \ldots, a_{2007}$ be real numbers such that

$$a_1 + a_2 + \cdots + a_{2007} \geq (2007)^2 \text{ and } a_1^2 + a_2^2 + \cdots + a_{2007}^2 \leq (2007)^3 - 1.$$

Prove that $a_k \in [2006; 2008]$ for all $k \in \{1, 2, \ldots, 2007\}$.

Question 10. What is the smallest possible value of

$$x^2 + 2y^2 - x - 2y - xy?$$

Question 11. Find all polynomials $P(x)$ satisfying the equation

$$(2x - 1)P(x) = (x - 1)P(2x), \ \ \forall x.$$

Question 12. Calculate the sum

$$\frac{1}{2.7.12} + \frac{1}{7.12.17} + \cdots + \frac{1}{1997.2002.2007}.$$

Question 13. Let ABC be an acute-angle triangle with $BC > CA$. Let O, H and F

be the circumcenter, orthocentre and the foot of its altitude CH,

respectively. Suppose that the perpendicular to OF at F meet the side

CA at P. Prove $\angle FHP = \angle BAC$.

Question 14. How many ordered pairs of integers (x, y) satisfy the equation

$$x^2 + y^2 + xy = 4(x + y)?$$

Question 15. Let $p = \overline{abcd}$ be the 4-digit prime number. Prove that the equation

$$ax^3 + bx^2 + cx + d = 0$$

has no rational roots.

1.3 Hanoi Open Mathematical Olympiad 2008

1.3.1 Junior Section

Question 1. How many integers from 1 to 2008 have the sum of their digits divisible
by 5 ?

Question 2. How many integers belong to $(a, 2008a)$, where a $(a > 0)$ is given.

Question 3. Find the coefficient of x in the expansion of

$$(1+x)(1-2x)(1+3x)(1-4x)\cdots(1-2008x).$$

Question 4. Find all pairs (m, n) of positive integers such that

$$m^2 + n^2 = 3(m+n).$$

Question 5. Suppose x, y, z, t are real numbers such that

$$\begin{cases} |x+y+z-t| \leqslant 1 \\ |y+z+t-x| \leqslant 1 \\ |z+t+x-y| \leqslant 1 \\ |t+x+y-z| \leqslant 1 \end{cases}$$

Prove that $x^2 + y^2 + z^2 + t^2 \leqslant 1$.

Question 6. Let $P(x)$ be a polynomial such that

$$P(x^2 - 1) = x^4 - 3x^2 + 3.$$

Find $P(x^2 + 1)$?

Question 7. The figure $ABCDE$ is a convex pentagon. Find the sum

$$\angle DAC + \angle EBD + \angle ACE + \angle BDA + \angle CEB?$$

Question 8. The sides of a rhombus have length a and the area is S. What is the length of the shorter diagonal?

Question 9. Let be given a right-angled triangle ABC with $\angle A = 90^0$, $AB = c$, $AC = b$. Let $E \in AC$ and $F \in AB$ such that $\angle AEF = \angle ABC$ and $\angle AFE = \angle ACB$. Denote by $P \in BC$ and $Q \in BC$ such that $EP \perp BC$ and $FQ \perp BC$. Determine $EP + EF + PQ$?

Question 10. Let $a, b, c \in [1, 3]$ and satisfy the following conditions

$$\max\{a, b, c\} \geqslant 2, \ a + b + c = 5.$$

What is the smallest possible value of

$$a^2 + b^2 + c^2?$$

1.3.2 Senior Section

Question 1. How many integers are there in $(b, 2008b]$, where b $(b > 0)$ is given.

Question 2. Find all pairs (m, n) of positive integers such that

$$m^2 + 2n^2 = 3(m + 2n).$$

Question 3. Show that the equation

$$x^2 + 8z = 3 + 2y^2$$

has no solutions of positive integers x, y and z.

Question 4. Prove that there exists an infinite number of relatively prime pairs (m, n) of positive integers such that the equation

$$x^3 - nx + mn = 0$$

has three distint integer roots.

Question 5. Find all polynomials $P(x)$ of degree 1 such that

$$\max_{a\leq x\leq b} P(x) - \min_{a\leq x\leq b} P(x) = b-a, \ \forall a,b\in\mathbb{R} \text{ where } a<b.$$

Question 6. Let $a,b,c\in[1,3]$ and satisfy the following conditions

$$\max\{a,b,c\}\geqslant 2,\ a+b+c=5.$$

What is the smallest possible value of

$$a^2+b^2+c^2?$$

Question 7. Find all triples (a,b,c) of consecutive odd positive integers such that $a<b<c$ and $a^2+b^2+c^2$ is a four digit number with all digits equal.

Question 8. Consider a convex quadrilateral $ABCD$. Let O be the intersection of AC and BD; M,N be the centroid of $\triangle AOB$ and $\triangle COD$ and P,Q be orthocenter of $\triangle BOC$ and $\triangle DOA$, respectively. Prove that $MN\perp PQ$.

Question 9. Consider a triangle ABC. For every point $M\in BC$ we difine $N\in CA$ and $P\in AB$ such that $APMN$ is a parallelogram. Let O be the intersection of BN and CP. Find $M\in BC$ such that $\angle PMO=\angle OMN$.

Question 10. Let be given a right-angled triangle ABC with $\angle A=90^0$, $AB=c$, $AC=b$. Let $E\in AC$ and $F\in AB$ such that $\angle AEF=\angle ABC$ and $\angle AFE=\angle ACB$. Denote by $P\in BC$ and $Q\in BC$ such that $EP\perp BC$ and $FQ\perp BC$. Determine $EP+EF+FQ$?

1.4 Hanoi Open Mathematical Olympiad 2009

1.4.1 Junior Section

Question 1. Let a, b, c be 3 distinct numbers from $\{1, 2, 3, 4, 5, 6\}$. Show that 7 divides $abc + (7 - a)(7 - b)(7 - c)$.

Question 2. Show that there is a natural number n such that the number $a = n!$ ends exacly in 2009 zeros.

Question 3. Let a, b, c be positive integers with no common factor and satisfy the conditions

$$\frac{1}{a} + \frac{1}{b} = \frac{1}{c}.$$

Prove that $a + b$ is a square.

Question 4. Suppose that $a = 2^b$, where $b = 2^{10n+1}$. Prove that a is divisible by 23 for any positive integer n.

Question 5. Prove that $m^7 - m$ is divisible by 42 for any positive integer m.

Question 6. Suppose that 4 real numbers a, b, c, d satisfy the conditions

$$\begin{cases} a^2 + b^2 = 4 \\ c^2 + d^2 = 4 \\ ac + bd = 2 \end{cases}$$

Find the set of all possible values the number $M = ab + cd$ can take.

Question 7. Let a, b, c, d be positive integers such that $a + b + c + d = 99$. Find the smallest and the greatest values of the following product $P = abcd$.

Question 8. Find all the pairs of the positive integers such that the product of the numbers of any pair plus the half of one of the numbers plus one third of the other number is three times less than 1004.

Question 9. Let be given ΔABC with $\text{ area }(\Delta ABC) = 60\text{cm}^2$. Let R, S lie in BC such that $BR = RS = SC$ and P, Q be midpoints of AB and AC, respectively. Suppose that PS intersects QR at T. Evaluate $\text{ area }(\Delta PQT)$.

Question 10. Let ABC be an acute-angled triangle with $AB = 4$ and CD be the altitude through C with $CD = 3$. Find the distance between the midpoints of AD and BC.

Question 11. Let $A = \{1, 2, \ldots, 100\}$ and B is a subset of A having 48 elements. Show that B has two distint elements x and y whose sum is divisible by 11.

1.4.2 Senior Section

Question 1. Let a, b, c be 3 distinct numbers from $\{1, 2, 3, 4, 5, 6\}$. Show that 7 divides $abc + (7 - a)(7 - b)(7 - c)$.

Question 2. Show that there is a natural number n such that the number $a = n!$ ends exacly in 2009 zeros.

Question 3. Let a, b, c be positive integers with no common factor and satisfy the conditions

$$\frac{1}{a} + \frac{1}{b} = \frac{1}{c}.$$

Prove that $a + b$ is a square.

Question 4. Suppose that $a = 2^b$, where $b = 2^{10n+1}$. Prove that a is divisible by 23 for any positive integer n.

Question 5. Prove that $m^7 - m$ is divisible by 42 for any positive integer m.

Question 6. Suppose that 4 real numbers a, b, c, d satisfy the conditions

$$\begin{cases} a^2 + b^2 = 4 \\ c^2 + d^2 = 4 \\ ac + bd = 2 \end{cases}$$

Find the set of all possible values the number $M = ab + cd$ can take.

Question 7. Let a, b, c, d be positive integers such that $a + b + c + d = 99$. Find the smallest and the greatest values of the following product $P = abcd$.

Question 8. Find all the pairs of the positive integers such that the product of the numbers of any pair plus the half of one of the numbers plus one third of the other number is three times less than 1004.

Question 9.Given an acute-angled triangle ABC with area S, let points A', B', C' be located as follows: A' is the point where altitude from A on BC meets the outwards facing semicirle drawn on BC as diameter. Points B', C' are located similarly. Evaluate the sum

$$T = (\text{area } \Delta BCA')^2 + (\text{area } \Delta CAB')^2 + (\text{area } \Delta ABC')^2.$$

Question 10. Prove that $d^2 + (a-b)^2 < c^2$, where d is diameter of the inscribed circle of ΔABC.

Question 11. Let $A = \{1, 2, \ldots, 100\}$ and B is a subset of A having 48 elements. Show that B has two distint elements x and y whose sum is divisible by 11.

1.5 Hanoi Open Mathematical Olympiad 2010

1.5.1 Junior Section

Question 1. Compare the numbers:

$$P = \underbrace{888\ldots888}_{2010\ digits} \times \underbrace{333\ldots333}_{2010\ digits} \text{ and } Q = \underbrace{444\ldots444}_{2010\ digits} \times \underbrace{666\ldots667}_{2010\ digits}$$

(A): $P = Q$; (B): $P > Q$; (C): $P < Q$.

Question 2. The number of integer n from the set $\{2000, 2001, \ldots, 2010\}$ such that $2^{2n} + 2^n + 5$ is divisible by 7:

(A): 0; (B): 1; (C): 2; (D): 3; (E) None of the above.

Question 3. 5 last digits of the number $M = 5^{2010}$ are

(A): 65625; (B): 45625; (C): 25625; (D): 15625; (E) None of the above.

Question 4. How many real numbers $a \in (1, 9)$ such that the corresponding number $a - \dfrac{1}{a}$ is an integer.

(A): 0; (B): 1; (C): 8; (D): 9; (E) None of the above.

Question 5. Each box in a 2×2 table can be colored black or white. How many different colorings of the table are there?

(A): 4; (B): 8; (C): 16; (D): 32; (E) None of the above.

Question 6. The greatest integer less than $(2+\sqrt{3})^5$ are

(A): 721; (B): 722; (C): 723; (D): 724; (E) None of the above.

Question 7. Determine all positive integer a such that the equation

$$2x^2 - 30x + a = 0$$

has two prime roots, i.e. both roots are prime numbers.

Question 8. If n and n^3+2n^2+2n+4 are both perfect squares, find n.

Question 9. Let be given a triangle ABC and points D, M, N belong to BC, AB, AC, respectively. Suppose that MD is parallel to AC and ND is parallel to AB. If $S_{\Delta BMD} = 9\text{cm}^2$, $S_{\Delta DNC} = 25\text{cm}^2$, compute $S_{\Delta AMN}$?

Question 10. Find the maximum value of

$$M = \frac{x}{2x+y} + \frac{y}{2y+z} + \frac{z}{2z+x}, \quad x, y, z > 0.$$

1.5.2 Senior Section

Question 1. The number of integers $n \in [2000, 2010]$ such that $2^{2n} + 2^n + 5$ is divisible by 7 is

(A): 0; (B): 1; (C): 2; (D): 3; (E) None of the above.

Question 2. 5 last digits of the number 5^{2010} are

(A): 65625; (B): 45625; (C): 25625; (D): 15625; (E) None of the above.

Question 3. How many real numbers $a \in (1, 9)$ such that the corresponding number $a - \dfrac{1}{a}$ is an integer.

(A): 0; (B): 1; (C): 8; (D): 9; (E) None of the above.

Question 4. Each box in a 2×2 table can be colored black or white. How many different colorings of the table are there?

Question 5. Determine all positive integer a such that the equation

$$2x^2 - 30x + a = 0$$

has two prime roots, i.e. both roots are prime numbers.

Question 6. Let a, b be the roots of the equation $x^2 - px + q = 0$ and let c, d be the roots of the equation $x^2 - rx + s = 0$, where p, q, r, s are some positive real numbers. Suppose that

$$M = \frac{2(abc + bcd + cda + dab)}{p^2 + q^2 + r^2 + s^2}$$

is an integer. Determine a, b, c, d.

Question 7. Let P be the common point of 3 internal bisectors of a given ABC. The line passing through P and perpendicular to CP intersects AC and BC at M and N, respectively. If $AP = 3$cm, $BP = 4$cm, compute the value of $\dfrac{AM}{BN}$?

Question 8. If n and $n^3 + 2n^2 + 2n + 4$ are both perfect squares, find n.

Question 9. Let x, y be the positive integers such that $3x^2 + x = 4y^2 + y$. Prove that $x - y$ is a perfect integer.

Question 10. Find the maximum value of

$$M = \frac{x}{2x+y} + \frac{y}{2y+z} + \frac{z}{2z+x}, \quad x, y, z > 0.$$

1.6 Hanoi Open Mathematical Olympiad 2011

1.6.1 Junior Section

Question 1. Three lines are drawn in a plane. Which of the following could NOT be the total number of points of intersections?

(A): 0; (B): 1; (C): 2; (D): 3; (E): They all could.

Question 2. The last digit of the number $A = 7^{2011}$ is

(A) 1; (B) 3; (C) 7; (D) 9; (E) None of the above.

Question 3. What is the largest integer less than or equal to

$$\sqrt[3]{(2011)^3 + 3 \times (2011)^2 + 4 \times 2011 + 5}?$$

(A) 2010; (B) 2011; (C) 2012; (D) 2013; (E) None of the above.

Question 4. Among the four statements on real numbers below, how many of them are correct?

"If $a < b < 0$ then $a < b^2$";
"If $0 < a < b$ then $a < b^2$";
"If $a^3 < b^3$ then $a < b$";
"If $a^2 < b^2$ then $a < b$";
"If $|a| < |b|$ then $a < b$".

(A) 0; (B) 1; (C) 2; (D) 3; (E) 4

Question 5. Let $M = 7! \times 8! \times 9! \times 10! \times 11! \times 12!$. How many factors of M are perfect squares?

Question 6. Find all positive integers (m, n) such that

$$m^2 + n^2 + 3 = 4(m + n).$$

Question 7. Find all pairs (x, y) of real numbers satisfying the system

$$\begin{cases} x + y = 3 \\ x^4 - y^4 = 8x - y \end{cases}$$

Question 8. Find the minimum value of

$$S = |x + 1| + |x + 5| + |x + 14| + |x + 97| + |x + 1920|.$$

Question 9. Solve the equation

$$1 + x + x^2 + x^3 + \cdots + x^{2011} = 0.$$

Question 10. Consider a right-angle triangle ABC with $A = 90^o$, $AB = c$ and $AC = b$. Let $P \in AC$ and $Q \in AB$ such that $\angle APQ = \angle ABC$ and $\angle AQP = \angle ACB$. Calculate $PQ + PE + QF$, where E and F are the projections of P and Q onto BC, respectively.

Question 11. Given a quadrilateral $ABCD$ with $AB = BC = 3$cm, $CD = 4$cm, $DA = 8$cm and $\angle DAB + \angle ABC = 180^o$. Calculate the area of the quadrilateral.

Question 12. Suppose that $a > 0, b > 0$ and $a + b \leqslant 1$. Determine the minimum value of

$$M = \frac{1}{ab} + \frac{1}{a^2 + ab} + \frac{1}{ab + b^2} + \frac{1}{a^2 + b^2}.$$

1.6.2 Senior Section

Question 1. An integer is called "octal" if it is divisible by 8 or if at least one of its digits is 8. How many integers between 1 and 100 are octal?

(A): 22; (B): 24; (C): 27; (D): 30; (E): 33.

Question 2. What is the smallest number

(A) 3; (B) $2^{\sqrt{2}}$; (C) $2^{1+\frac{1}{\sqrt{2}}}$; (D) $2^{\frac{1}{2}} + 2^{\frac{2}{3}}$; (E) $2^{\frac{5}{3}}$.

Question 3. What is the largest integer less than to

$$\sqrt[3]{(2011)^3 + 3 \times (2011)^2 + 4 \times 2011 + 5}?$$

(A) 2010; (B) 2011; (C) 2012; (D) 2013; (E) None of the above.

Question 4. Prove that

$$1 + x + x^2 + x^3 + \cdots + x^{2011} \geqslant 0$$

for every $x \geqslant -1$.

Question 5. Let a, b, c be positive integers such that $a + 2b + 3c = 100$. Find the greatest value of $M = abc$.

Question 6. Find all pairs (x, y) of real numbers satisfying the system

$$\begin{cases} x + y = 2 \\ x^4 - y^4 = 5x - 3y \end{cases}$$

Question 7. How many positive integers a less than 100 such that $4a^2 + 3a + 5$ is divisible by 6.

Question 8. Find the minimum value of

$$S = |x+1| + |x+5| + |x+14| + |x+97| + |x+1920|.$$

Question 9. For every pair of positive integers $(x;y)$ we define $f(x;y)$ as follows:

$$\begin{aligned} &f(x,1) = x \\ &f(x,y) = 0 \quad \text{if} \quad y > x \\ &f(x+1,y) = y[f(x,y) + f(x,y-1)] \end{aligned}$$

Evaluate $f(5;5)$.

Question 10. Two bisectors BD and CE of the triangle ABC intersect at O. Suppose that $BD.CE = 2BO.OC$. Denote by H the point in BC such that $OH \perp BC$. Prove that $AB.AC = 2HB.HC$.

Question 11. Consider a right-angle triangle ABC with $A = 90^o$, $AB = c$ and $AC = b$. Let $P \in AC$ and $Q \in AB$ such that $\angle APQ = \angle ABC$ and $\angle AQP = \angle ACB$. Calculate $PQ + PE + QF$, where E and F are the projections of P and Q onto BC, respectively.

Question 12. Suppose that $|ax^2 + bx + c| \geqslant |x^2 - 1|$ for all real numbers x. Prove that $|b^2 - 4ac| \geqslant 4$.